MATH 215: CHAPTER 5- FUNCTIONS

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1. ABSTRACT FUNCTIONS

Functions are among the most common mathematical objects and appears in almost every mathematical theory. Intuitively speaking, a function is just a machine which attaches to **every** element a (the input) in a give set A (the domain of the function) a **unique** element $f(a)$ (the output/ the image of a) in a set B (the range of the function). To illustrate these ideas, here are some day-to-day examples:

(1) The function which attaches to every person its height. The domain of the function is the set of humans and the range of the functions is the set of real numbers (theoretically, a person can $\sqrt{2}$-foot tall).
(2) If we attaches to every person, its siblings, the result is not a function and there are two reason for that. The first is that there are people with no siblings (and therefore the function is not defined for **every** person), also there are people with more than one sibling and for those people, we do not attach a **unique** person).

To formal way to define a function is as sets of pairs:

Definition 1.1. Let A, B be two sets. A *function* from A to B is a set of orderes pairs $f \subseteq A \times B$ such that:

(1) f is total *Total* on A: $\forall a \in A. \exists b \in B. \langle a, b\rangle \in f$.
(2) f is *univalent*: $\forall a \in A. \forall b_1, b_2 \in B. \langle a, b_1\rangle \in f \wedge \langle a, b_2\rangle \in f \Rightarrow b_1 = b_2$.

Notation 1.2. If f is a function from A to B we denote it by $f : A \to B$. Also if $f : A \to B$ is a function, we denote $f(a) = b$ if and only if $\langle a, b\rangle \in f$. So $f(a)$ is the unique object in the set B that the function f attaches to the element a.

Example 1.3. (1) Let $f = \{\langle 1, a\rangle, \langle 3, b\rangle, \langle 2, a\rangle\}$. To see that f is a function from $\{1, 2, 3\}$ to $\{a, b, c\}$, we need to prove that for every $x \in \{1, 2, 3\}$ the is a unique $y \in \{a, b, c\}$ such that $\langle x, y\rangle \in f$ (and then we can denote $f(x) = y$). Since there are only 3 elements in f we can go one-by-one over the elements of f and check that this is indeed the case manually. Now that we are sure that f is a function, we can write $f : \{1, 2, 3\} \to \{a, b\}$ and

$$f(1) = a,\ f(2) = a,\ f(3) = b.$$

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(2) The identity relation on a set A, is a function $id_A : A \to A$ satisfying $id_A(a) = a$ for every $a \in A$.

(3) Consider $S = \{\langle X, x\rangle \in P(\mathbb{N}) \times \mathbb{N} \mid x \in X\}$. This is **not** a function from $P(\mathbb{N})$ to $\mathbb{N}$ since it is not total. For example[1], $\emptyset \in P(\mathbb{N})$, and there is no x such that $\langle \emptyset, x\rangle \in S$, otherwise we would have $x \in \emptyset$. Let us try and remove $\emptyset$ to see if we get a function. Is S a function from $P(\mathbb{N}) \setminus \{\emptyset\}$ to $\mathbb{N}$? This is still not a function since it is not univalent. For example, $\langle \{1, 2, 3\}, 1\rangle, \langle \{1, 2, 3\}, 2\rangle \in S$. Also it is not $Total$

(4) Let A, B be any sets. For every $b \in B$ the *constant function with value* b is the the relation f_b from A to B

$$f_b = \{\langle x, b\rangle \mid x \in A\} = A \times \{b\}.$$

Claim: f_b is a function from A to B.

Proof. We need to prove that f_b is total on A and univalent.
<u>Total:</u> We need to prove that for every $x \in A$ there is $y \in B$ such that $\langle x, y\rangle \in f_b$. Let $x \in A$. Define $y = b$, then by the definition of f_b, $\langle x, b\rangle \in f_b$.
<u>Univalent</u>: We need to prove that for every $a \in A$ and for every $b_1, b_2 \in B$, if $\langle a, b_1\rangle, \langle a, b_2\rangle \in f_b$ then $b_1 = b_2$. Let $a \in A$, $b_1, b_2 \in B$ and suppose that $\langle a, b_1\rangle, \langle a, b_2\rangle \in f_b$. We want to prove that $b_1 = b_2$. By the definition of f_b, since we have that $b_1 = b = b_2$. $\square$

Hence $f_b : A \to B$ is a function satisfying $\forall a \in A. f_b(a) = b$.

(5) $\pi_1 : A \times B \to A$ $\pi_1 = \{\langle \langle a, b\rangle, c\rangle \in (A \times B) \times A \mid a = c\}$ Is called the *projection to the left coordinate*, it satisfies that $\pi(\langle a, b\rangle) = a$. Similarly, the *projection to the right coordinate* is denoted $\pi_2 : (A \times B) \to B$ and it satisfies $\pi_2(\langle a, b\rangle) = b$.

(6) To summation operation on the rational number (or on the natural numbers/integers/reals) is a function $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We are used to writing $3 + 5 = 8$ instead of $+(\langle 3, 5\rangle) = 8$.

(7) Let $g : P(A) \times P(B) \to P(A)$ defined by $g = \{\langle\langle X, Y\rangle, Z\rangle \in (P(A) \times P(B)) \times P(A) \mid Z = X \cap Y\}$ we have that $g(X, Y) = X \cap Y$

Definition 1.4. A sequence of elements in the set A is a function $f : \mathbb{N} \to A$. In calculus we sometime denote $a_n = f(n)$ and $(a_n)_{n=0}^{\infty} = f$.

Example 1.5. The sequence $1, \frac{1}{2}, \frac{1}{3}, ...$ is formally the function $f : \mathbb{N} \to \mathbb{Q}$, $f = \{\langle n, \frac{1}{n+1}\rangle \mid n \in \mathbb{N}\}$ satisfying $f(n) = \frac{1}{n+1}$.

Definition 1.6. Let $f : A \to B$ be a function. The *domain* of f is simply A, we denote $\mathrm{dom}(f) = A$. The *range* of f is B and we denote $\mathrm{Range}(f) = B$. The *image* of f is the set $\mathrm{Im}(f) = \{f(a) \mid a \in A\}$.

Note that $\mathrm{Im}(f) \subseteq \mathrm{Range}(f)$.

[1] To prove that e function is not total/univalent, we should provide a counter example.

Example 1.7. For the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2$ we have that $\mathrm{dom}(f) = \mathrm{Range}(f) = \mathbb{R}$ while $\mathrm{Im}(f) = [0, \infty)$. Since the last equality if a set equality, we should prove it by a double implication:

(1) $\subseteq$: Let $r \in \mathrm{Im}(f)$, we need to prove that $r \in [0, \infty)$. By definition of $\mathrm{Im}(f)$, there is $x \in \mathbb{R}$ such that $f(x) = r$. Those $r = x^2 \geq 0$ and by definition of $[0, \infty)$, $r \in [0, \infty)$.
(2) $\supseteq$: Let $r \in [0, \infty)$. we need to prove that $r \in \mathrm{Im}(f)$. By definition, $r \geq 0$ and therefore we have $\sqrt{r}$ defined. Define (This is an existential proof) $x = \sqrt{r}$, then $f(x) = x^2 = r$.

Definition 1.8. Let A, B be two sets. We denote *the set of all functions* from A to B by

$$^{A}B = \{f \in P(A \times B) \mid f \text{ is a function from } A \text{ to } B\}$$

Example 1.9. Let F_2 be the relation from $^{\mathbb{R}}\mathbb{R}$ to $\mathbb{R}$ defined by

$$F_2 = \{\langle f, r\rangle \in {}^{\mathbb{R}}\mathbb{R} \times \mathbb{R} \mid \langle 2, r\rangle \in f\}.$$

Prove that F is a function.

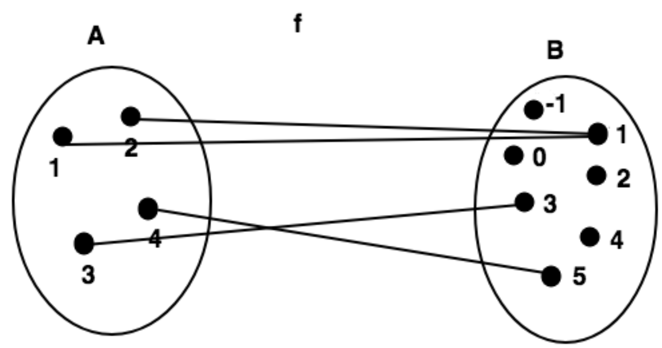
Proof. Total: We nee to probe that for every $f \in \mathbb{R}\mathbb{R}$ (here the domain of F_2 is itself a set of functions!) there is $r \in \mathbb{R}$ such that $\langle f, r\rangle \in F$. Let $f \in \mathbb{R}\mathbb{R}$. we need to find $r \in \mathbb{R}$ such that $\langle 2, r\rangle \in f$. Since f is a function from $\mathbb{R}$ to $\mathbb{R}$, it is in particular a total relation on $\mathbb{R}$, and since $2 \in \mathbb{R}$, there exists $r \in \mathbb{R}$ such that $\langle 2, r\rangle \in f$, hence $\langle f, r\rangle \in F_2$.
Univalent: We want to prove that for any $f \in {}^{\mathbb{R}}\mathbb{R}$ and any $r_1, r_2 \in \mathbb{R}$, if $\langle f, r_1\rangle, \langle f, r_2\rangle \in F_2$ then $r_1 = r_2$. Supposet that $\langle f, r_1\rangle, \langle f, r_2\rangle \in F_2$, then by definition $\langle 2, r_1\rangle, \langle 2, r_2\rangle \in f$. Since f is a function, it is in particular univalent and therefore $r_1 = r_2$. $\square$

Note that we have $F_2(f) = f(2)$ for every function $f \in {}^{\mathbb{R}}$.

2. How to work with functions

Given a set of pairs R in $A \times B$ we can represent R as a collections of arrows from he set A to the set B. This is very convenient when considering functions. For example, to verify the R is a function from A to B we should simply verify(not prove!) that there is **exactly one** arrow attached to **every** element of A. For example, consider

$$f : \{1, 2, 3, 4\} \to \{-1, 0, 1, 2, 3, 4, 5\} \;\; f = \{\langle 1, 1\rangle, \langle 2, 1\rangle, \langle 3, 3\rangle \langle 4, 5\rangle\}$$



In order to discard the need to formulate functions as sets of pair we simply need to understand when two functions are equal[2]. This is given by the following theorem which we omit the proof.

Theorem 2.1. *Let f, g be two function. Then the following are equivalent:*

(1) $\mathrm{dom}(f) = \mathrm{dom}(g)$ *and* $\forall x \in \mathrm{dom}(f). f(x) = g(x)$.
(2) $f = g$.

The theorem says that two functions are equal if and only if the functions have the same domain and to every x in this domain, the function attach then same element in their range. So in order to describe a function, we simply have to say what is the domain and describe for every element in the domain where the function maps/sends it. Here are some of the most common ways to define functions in this way:

(1) Defining a function with a formula: The definition has the form " Define $f : A \to B$ by $f(a) =$(some formula)". For example, we can define $f : \mathbb{R} \to \mathbb{R}$ by $f(r) = 2$, this is the constant function which for every real r returns the value 2. Another example, define $g : P(\mathbb{N}) \to P(\mathbb{N})$ by $g(X) = X \cup \{1, 2\}$. Then for example $g(\{1, 3, 4\}) = \{1, 2, 3, 4\}$ and $g(\mathbb{N}) = \mathbb{N}$.

Important: If we define $f : A \to B$ by a formula $f(a) =$(some formula) we **must** always make sure that the functions we define are well defined in the sense that:

(a) The function is total. Practically, this means that we should make sure that the formula for $f(a)$ is defined for every $a \in A$.
(b) The function is univalent. This means that for every $a \in A$, the formula for $f(a)$ points to a single element. (This is trivial in most cases)

[2] As we did with tuples.

(c) for every $a \in A$ the formula for $f(a)$ returns an element in B. So the ranged we declared when we wrote $f : A \to B$ is indeed correct.

(2) Definition of a function by cases: Suppose we which to define a function on a set A, and for some of the elements of A we want one formula and for the another part of A we want to use a different formula. We can do that the following way: "Define $f : A \to B$ by

$$f(a) = \begin{cases} \text{(first formula)} & \text{(first condition on } a) \\ \text{(second formula)} & \text{(second condition on a)} \\ \ldots \end{cases}$$

where the conditions on a describe the element for which you would like to use the formula. When we check that a function defined by cases is well defined, we also have to check the the condition on a covers all possible a and that they are "disjoint" in the sense that no a satisfy two of the condition. We can also use "otherwise" if we would like to take care of the remaining cases

Remark 2.2. The function equality theorem indicated that a function is **not** the same as a formula defining it.

For example the functions: $f_1, f_2 : \{-1, 0, 1\} \to \mathbb{R}$ defined by $f_1(x) = |x|$ and $f_2(x) = x^2$ have different formulas but they define the same function since $f_1(-1) = f_2(-1),\ f_1(0) = f_2(0),\ f_1(1) = f_2(1)$.

Remember! Different formulas can define the same function.

(1) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$ satisfies $f(4) = 16$.
(2) $g : \mathbb{N} \to P(\mathbb{N})$ defined by $g(x) = \{x, x+1\}$ satisfies $g(5) = \{5, 6\}$.
(3) $t : \mathbb{N} \to \mathbb{N}$ defined by $t(n) = \begin{cases} 0 & n \in \mathbb{N}_{even} \\ 1 & n \in \mathbb{N}_{odd} \end{cases}$.
satisfies that $t(1) = 1, t(14) = 0$. $\ s(f)(3) = \{-2\}$.
(4) $F : P(\mathbb{N})^2 \to \mathbb{N}$ defined by $F(\langle A, B\rangle) = \begin{cases} 0 & A \cap B = \emptyset \\ \min(A \cap B) & else \end{cases}$
satisfies that $F(\langle \{1, 2, 3, 4\}, \mathbb{N}_{even}\rangle) = 2$.
(5) $f : \mathbb{N}^2 \to P(\mathbb{N})$ defined by $f(\langle x, y\rangle) = \{n \in \mathbb{N} \mid x < n < y\}$ satisfies $f(\langle 1, 4\rangle) = \{2, 3\}$ -$f(\langle 4, 1\rangle) = \emptyset$.

Definition 2.3. Let $f : A \to B$ be a function and $X \subseteq A$. We define the *restriction of* f *to* X, denote by $f \restriction X : X \to B$, and a function with domain $dom(f \restriction X) = X$ and for every $x \in X$, $(f \restriction X)(x) = f(x)$.

Intuitively, the restriction of a function acts the same way that the original function did, the only difference is that the domain restricts to the new set X.

Definition 2.4. Let A be any set. We define the *Identity function* on A as the function $Id_A : A \to A$ defined by $Id_A(a) = a$.

Example 2.5. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as $f(z) = |z|$. Prove that $f \restriction \mathbb{N} = Id_{\mathbb{N}}$

Proof. We want to prove equality of functions. First we want to prove that $\text{dom}(f \restriction \mathbb{N}) = \text{dom}(Id_{\mathbb{N}})$. Indeed by definition of restriction and the identity function, both of the functions have domain $\mathbb{N}$. Next we want to prove that $\forall x \in \mathbb{N}.(f \restriction \mathbb{N})(x) = Id_{\mathbb{N}}(x)$. Let $x \in \mathbb{N}$, then by definition of restriction and since $n \geq 0$ we have

$$(f \restriction \mathbb{N})(x) = f(x) = |x| = x$$

and by definition of the identity function we have

$$Id_{\mathbb{N}}(x) = x$$

Hence

$$(f \restriction \mathbb{N})(x) = x = Id_{\mathbb{N}}(x)$$

as wanted □

Definition 2.6. Let $f : A \to B$ and $g : B \to C$ be two functions. We define the composition of g in f as $g \circ f : A \to C$, to be the function with domain f and range C such that for each $a \in A$, $(g \circ f)(a) = g(f(a))$.

Example 2.7.

3. Properties of functions

Definition 3.1. Let $f : A \to B$ be a function we sat that f is:

(1) One to one/ injective: if for every $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$.
(2) Onto/ surjective: if for every $b \in B$ there is $a \in A$ such that $f(a) = b$.

Example 3.2. (1) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not injective as $1 \neq -1$ and $f(-1) = (-1)^2 = 1 = 1^2 = f(1)$.
(2) $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = n - 1$ is injective.

Proof. Let $n_1, n_2 \in \mathbb{N}$. Suppose that $f(n_1) = f(n_2)$, we want to prove that $n_1 = n_2$. By definition of f, $n_1 - 1 = n_2 - 1$, adding 1 to both sides of the equation we conclude that $n_1 = n_2$. □

(3) $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined by $g(\langle n, m \rangle) = \langle 2n + m, n + m$ is injective.

Proof. Let $\langle n_1, m_1 \rangle, \langle n_2, m_2 \rangle \in \mathbb{N} \times \mathbb{N}$ and assume that $g(\langle n_1, m_1 \rangle) = g(\langle n_2, m_2 \rangle)$ we want to prove that $\langle n_1, m_1 \rangle = \langle n_2, m_2 \rangle$. By the assumption we know that $\langle 2n_1 + m_1, n_1 + m_1 = \langle 2n_2 + m_2, n_2 + m_2$ and by equality of pair we get that

$$2n_1 + m_1 = 2n_2 + m_2 \text{ and } n_1 + m_1 = n_2 + m_2$$

Subtracting the second equation from the first we get:

$$2n_1 + m_1 - (n_1 + m_1) = 2n_2 + m_2 - (n_2 - m_2)$$

$$n_1 = n_2$$

Hence by the equality $n_1 + m_1 = n_2 + m_2$, we have that $n_1 = n_2$ cancels so $m_1 = m_2$. By equality of pairs we conclude that $\langle n_1, m_1\rangle = \langle n_2, m_2\rangle$. □

(4) $F : P(\mathbb{N}) \to \mathbb{N})$ defined by $F(X) = \{x + 1 \mid x \in X\}$ is injective.

Proof. Let $X_1, X_2 \in P(\mathbb{N})$, suppose that $F(X_1) = F(X_2)$ we want to prove that $X_1 = X_2$. By definition of F,

$$)*) \quad \{x + 1 \mid x \in X_1\} = \{x + 1 \mid x \in X_2\}$$

Let us prove $X_1 = X_2$ by a double inclusion:

(a) $\underline{X_1 \subseteq X_2}$: Let $x_0 \in X_1$ we want to prove that $x_0 \in X_2$. By definition $x_0+1 \in \{x+1 \mid x \in X_1\}$ and by $(*)$, $x_0+1 \in \{x+1 \mid x \in X_2\}$. By the replacement principle, there exists $y \in X_2$ such that $x_0+1 = y+1$, hence $x_0 = y \in X_2$, which implies that $x_0 \in X_2$ as wanted.

(b) $\underline{X_2 \subseteq X_1}$: Symmetric to the first inclusion.

□

(5) $F_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $F(\langle n, m\rangle) = 2^n \cdot 3^m$ is injective.

Proof. Let $\langle n_1, m_1, \langle n_2, m_2\rangle \in N \times \mathbb{N}$. Suppose that $F_1(n_1, m_1) = F_1(n_2, m_2)$ we want to prove that $\langle n_1, m_1\rangle = \langle n_2, m_2\rangle$. By definition of F_2 we have that $(*)\quad 2^{n_1}3^{m_1} = 2^{n_2}3^{m_2}$. By the fundamental theorem of arithmentics, each positive natural number has a unique factorization into primes. The equality $(*)$ provides two factorization into primes of the same numbers, hence it must be the same, namely $n_1 = n_2$ and $m_1 = m_2$. By the basic property of pairs, $\langle n_1, m_1\rangle = \langle n_2, m_2\rangle$. □

Definition 3.3. Let $f : A \to B$ be a function. The image of f, denoted by $Im(f) = \{f(x) \mid x \in A\}$.

Note that f is surjective if and only if $Im(f) = Range(f)$.

Example 3.4. (1) The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$ is not surjective.

Proof. For example $1 \in \mathbb{N}$ and for every $n \in \mathbb{N}$, $f(n) \neq 1$. Otherwise, there exists $n \in \mathbb{N}$ such that $f(n) = 1$ then by definition of f, $2n = 1$ which implies that 1 is even, contradiction. □

Note also that $Im(f) = \mathbb{N}_{even}$ and that f is injective.

(2) The function $g : P(\mathbb{Z}) \to P(\mathbb{N})$ defined by $g(X) = X \cap \mathbb{Z}$ is surjective.

Proof. Let $Y \in P(\mathbb{N})$ we want to prove that there is $X \in P(\mathbb{Z})$ such that $f(X) = Y$. Define $X = Y$, then since $Y \in P(\mathbb{N})$, $Y \in P(\mathbb{Z})$. Also, to see that $g(Y) = Y$, we need to prove that $Y \cap \mathbb{N} = Y$. This is equivalent (by a proposition we have seen previously) to the fact that $Y \subseteq \mathbb{N}$. This follows since $Y \subseteq \mathbb{N}$. □

Also note that $Im(g) = P(\mathbb{N})$, (since we just proved that g is surjective) and it is not injective since for example $g(\{-1, 1\}) = \{1\} = g(\{1\})$.

(3) The function $h : (0,\infty) \to (0,\infty)$ defined by $h(x) = \frac{1}{x}$ is surjective.
Proof. Let $y \in (0,\infty)$, we want to prove that there is $x \in (0,\infty)$ such that $h(x) = y$. Namely, we want that $\frac{1}{x} = y$. Then define $x = \frac{1}{y}$. Since $0 < y$, also $0 < x$ and therefore $x \in (0,\infty)$ and we have that $h(x) = \frac{1}{\frac{1}{y}} = y$ as wanted. □

Proposition 3.5. *Let $f : A \to B$ and $g : B \to C$ be any functions.*
(1) If f, g are injective then so is $g \circ f$.
(2) If f, g are surjective then so is $g \circ f$

Definition 3.6. A function $f : A \to B$ is invertible if there is a function $g : B \to A$ such that:

$$g \circ f = id_A \quad \text{and } f \circ g = id_B$$

Example 3.7.

Theorem 3.8. *If g_1, g_2 are two inverse functions of f then $g_1 = g_2$. We denote the inverse function of f by f^{-1}.*

Proof. Suppose the g_1, g_2 are two inverse function of f, then

$$g_1 \circ f = id_A \quad \text{and } f \circ g_1 = id_B$$
$$g_2 \circ f = id_A \quad \text{and } f \circ g_2 = id_B$$

It follows that

$$g_1 = g_1 \circ Id_B = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = Id_A \circ g_2 = g_2$$

□

Theorem 3.9. *A function $f : A \to B$ is invertible if and only if it is one to one and onto.*

Proof. Suppose that f is invertible and let $f^{-1} : B \to A$ be the inverse function. Let us prove that f is one to one and onto:

- one to one: Let $a_1, a_2 \in A$, suppose that $f(a_1) = f(a_2)$, we want to prove that $a_1 = a_2$. Then $f^{-1}(f(a_1)) = f^{-1}(f(a_2))$ and since $f^{-1} \circ f = Id_A$ we get that
$$a_1 = f^{-1}(f(a_1)) = f^{-1}(f(a_2)) = a_2$$
- onto: Let $b \in B$, we want to prove that there is $a \in A$ such that $f(a) = b$. Let $a = f^{-1}(b) \in A$. Then $f(a) = f(f^{-1}(b))$ and since $f \circ f^{-1} = Id_B$, we have that $f(a) = f(f^{-1}(b)) = b$ as wanted.

For the other direction, suppose that f is one to one and onto B. We want to prove that f is invertible, namely that there is a function $g : B \to A$ such that $f \circ g = Id_B$ and $g \circ f = Id_A$. Here is the definition of g: For any element of b, there is (since f is onto B) a unique (since f is one to one) element $a_b \in A$ such that $f(a_b) = b$. Define $g(b) = a_b$. Let us prove that g is inverse to f:

- $\underline{g \circ f = Id_A}$: Let $a \in A$, then denote $f(a) = b \in B$. By definition $g(b) = a_b$ is the unique element in A such that $f(a_b) = b$ and since $f(a) = b$ it follows that $a = a_b$. Hence $g(f(a)) = g(b) = a_b = a$. It follows that $g \circ f = Id_A$.
- $\underline{f \circ g = Id_B}$: Let $b \in B$, by definition, $g(b) = a_b$ and a_b has the property that it is (the unique which is) mapped to b, namely $f(a_b) = b$. Hence $f(g(b)) = f(a_b) = b$. Again it follows that $f \circ g = Id_B$.

$\square$